Lecture 5. The equations of lines and surfaces in the space of the plane equation.

An equation of straight line going through two different points (x_1, y_1, z_1) and (x_2, y_2, z_2):

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}, \quad \text{at } x_1 \neq x_2,\ y_1 \neq y_2,\ z_1 \neq z_2 .$$

A parametric equation of straight line, going through a point (x_0, y_0, z_0) and parallel to a *direction vector* (a, b, c) of a straight line:

$$\begin{cases} x = x_0 + at, \\ y = y_0 + bt, \\ z = z_0 + ct, \end{cases} \quad t \in R .$$

Let two planes $Ax+By+Cz+D=0$ and $Ex+Fy+Gz+H=0$ be given, moreover, their normal vectors aren't collinear, then the set of simultaneous equations

$$\begin{cases} Ax+By+Cz+D=0, \\ Ex+Fy+Gz+H=0, \end{cases}$$

describes ***an intersection line*** of these planes.

Let (a, b, c) and (p, q, r) be direction vectors of two straight lines, then ***a parallelism condition of straight lines*** is :

$$aq - bp = br - cq = ar - cp = 0 ,$$

a perpendicularity condition of straight lines is :

$$ap + bq + cr = 0 ,$$

an angle α ***between straight lines:***

$$\cos\alpha = \frac{|ap+bq+cr|}{\sqrt{a^2+b^2+c^2}\ \sqrt{p^2+q^2+r^2}} ,$$

an angle α ***between a straight line and a plane:***

$$\cos\alpha = \frac{|Aa+Bb+Cc|}{\sqrt{a^2+b^2+c^2}\ \sqrt{A^2+B^2+C^2}} .$$

An equation of sphere of a radius R with a center in the point (a, b, c) is :

$$(x-a)^2+(y-b)^2+(z-c)^2=R^2$$

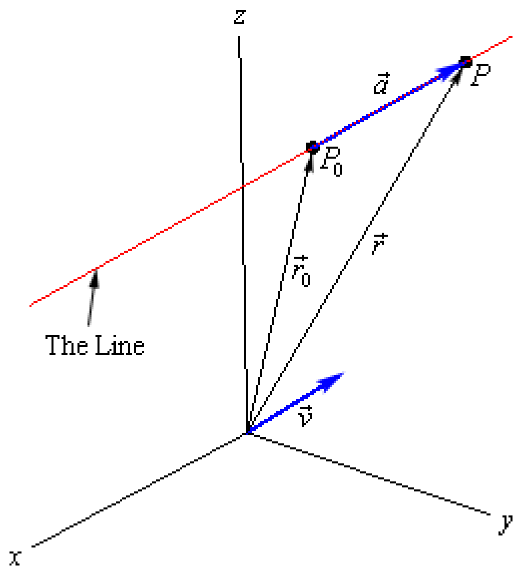
Equations of Lines **Paul's Online Math Notes**

The equation $y = mx + b$ does not describe a line in $\mathbb{R}^3$, instead it describes a plane.

Suppose that we know a point that is on the line, $P_0 = (x_0, y_0, z_0)$, and that $\vec{v} = \langle a, b, c \rangle$ is some vector that is parallel to the line. Note, in all likelihood, $\vec{v}$ will not be on the line itself. We only need $\vec{v}$ $\vec{v}$ to be parallel to the line. Finally, let $P = (x, y, z)$ be any point on the line.

Now, since our "slope" is a vector let's also represent the two points on the line as vectors. We'll do this with position vectors. So, let $\vec{r_0}$ and $\vec{r}$ be the position vectors for P_0 and P respectively. Also, for no apparent reason, let's define $\vec{a}$ to be the vector with representation $\overrightarrow{P_0 P}$.

We now have the following sketch with all these points and vectors on it.



Now, we've shown the parallel vector, $\vec{v}$, as a position vector but it doesn't need to be a position vector. It can be anywhere, a position vector, on the line or off the line, it just needs to be parallel to the line.

Next, notice that we can write $\vec{r}$ as follows,

$$\vec{r} = \vec{r_0} + \vec{a}$$

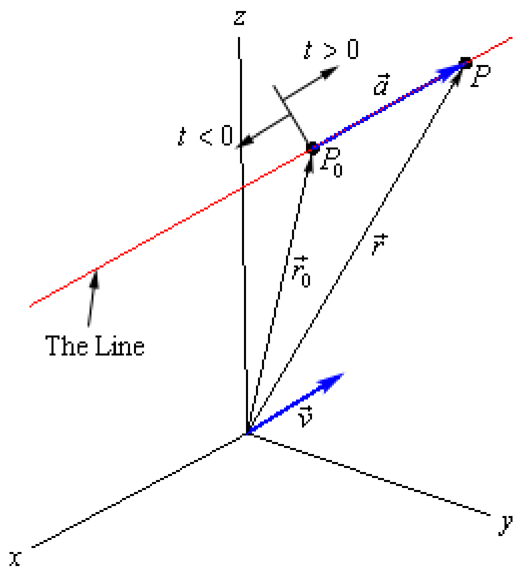
If you're not sure about this go back and check out the sketch for vector addition in the **vector arithmetic** section. Now, notice that the vectors $\vec{a}$ and $\vec{v}$ are parallel. **Therefore** there is a number, t, such that

$$\vec{a} = t\vec{v}$$

We now have,

$$\vec{r} = \vec{r_0} + t\vec{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

This is called the **vector form of the equation of a line**. The only part of this equation that is not known is the *t*. Notice that $t\vec{v}$ will be a vector that lies along the line and it tells us how far from the original point that we should move. If *t* is positive we move away from the original point in the direction of $\vec{v}$ (right in our sketch) and if *t* is negative we move away from the original point in the opposite direction of $\vec{v}$ (left in our sketch). As *t* varies over all possible values we will completely cover the line. The following sketch shows this dependence on *t* of our sketch.



There are several other forms of the equation of a line. To get the first alternate form let's start with the vector form and do a slight rewrite.

$$\vec{r} = \langle x_0, y_0, z_0 \rangle + t\langle a$$

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb,$$

The only way for two vectors to be equal is for the components to be equal. In other words,

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

This set of equations is called the **parametric form of the equation of a line**. Notice as well that this is really nothing more than an extension of the **parametric equations** we've seen previously. The only difference is that we are now working in three dimensions instead of two dimensions.

To get a point on the line all we do is pick a *t* and plug into either form of the line. In the vector form of the line we get a position vector for the point and in the parametric form we get the actual coordinates of the point.

There is one more form of the line that we want to look at. If we assume that *a*, *b*, and *c* are all non-zero numbers we can solve each of the equations in the parametric form of the line for *t*. We can then set all of them equal to each other since *t* will be the same number in each. Doing this gives the following,

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

This is called the **symmetric equations of the line**.

If one of *a*, *b*, or *c* does happen to be zero we can still write down the symmetric equations. To see this let's suppose that $b=0$. In this case *t* will not exist in the parametric equation for *y* and so we will only solve the parametric equations for *x* and *z* for *t*. We then set those equal and acknowledge the parametric equation for *y* as follows,

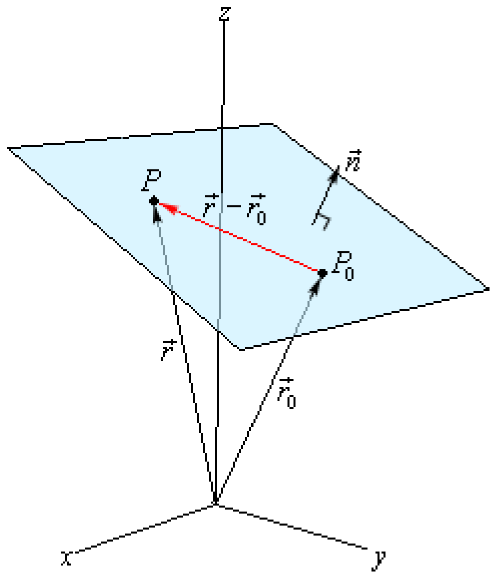
$$\frac{x-x_0}{a}=\frac{z-z_0}{c} \qquad y=y_0$$

Equations of Planes

In the first section of this chapter we saw a couple of equations of planes. However, none of those equations had three variables in them and were really extensions of graphs that we could look at in two dimensions. We would like a more general equation for planes.

So, let's start by assuming that we know a point that is on the plane, $P_0=\left(x_0,y_0,z_0\right)$. Let's also suppose that we have a vector that is orthogonal (perpendicular) to the plane, $\vec{n}=\left\langle a,b,c\right\rangle$. This vector is called the **normal vector**. Now, assume that $P=\left(x,y,z\right)$ is any point in the plane. Finally, since we are going to be working with vectors initially we'll let $\vec{r}_0$ and $\vec{r}$ be the position vectors for P_0 and *P* respectively.

Here is a sketch of all these vectors.



Notice that we added in the vector $\vec{r}-\vec{r}_0$ which will lie completely in the plane. Also notice that we put the normal vector on the plane, but there is actually no reason to expect this to be the case. We put it here to illustrate the point. It is completely possible that the normal vector does not touch the plane in any way.

Because $\vec{n}$ is orthogonal to the plane, it's also orthogonal to any vector that lies in the plane. In particular it's orthogonal to $\vec{r}-\vec{r}_0$. Recall from the **Dot Product** section that two orthogonal vectors will have a dot product of zero. In other words,

$$\vec{n}\bullet\left(\vec{r}-\vec{r}_0\right)=0 \qquad \Rightarrow$$

$$\vec{n}\bullet\left(\vec{r}-\vec{r}_0\right)=0 \qquad \Rightarrow \qquad \vec{n}\bullet$$

This is called the **vector equation of the plane**.

A slightly more useful form of the equations is as follows. Start with the first form of the vector equation and write down a vector for the difference.

$$\langle a,b,c\rangle\bullet\left(\langle x,y,z\rangle-\langle x_0,y_0,\right.$$

$$\langle a,b,c\rangle\bullet\langle x-x_0,y-y_0,z-$$

Now, actually compute the dot product to get,

$$a\left(x-x_0\right)+b\left(y-y_0\right)+c\left(z-z_0\right)=0$$

This is called the **scalar equation of plane**. Often this will be written as,

$$ax+by+cz=d$$

where $d = ax_0 + by_0 + cz_0$

This second form is often how we are given equations of planes. Notice that if we are given the equation of a plane in this form we can quickly get a normal vector for the plane. A normal vector is,

$$\vec{n} = \langle a, b, c \rangle$$

Quadric Surfaces

Quadric surfaces are the graphs of any equation that can be put into the general form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz +$$

where *A*, … , *J* are constants.

There is no way that we can possibly list all of them, but there are some standard equations so here is a list of some of the more common quadric surfaces.

Ellipsoid

Here is the general equation of an ellipsoid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Here is a sketch of a typical ellipsoid.

If $a = b = c$ then we will have a sphere.

Notice that we only gave the equation for the ellipsoid that has been centered on the origin. Clearly ellipsoids don't have to be centered on the origin. However, in order to make the discussion in this section a little easier we have chosen to concentrate on surfaces that are "centered" on the origin in one way or another.

Cone

Here is the general equation of a cone.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Here is a sketch of a typical cone.

Note that this is the equation of a cone that will open along the *z*-axis. To get the equation of a cone that opens along one of the other axes all we need to do is make a slight modification of the equation. This will be the case for the rest of the surfaces that we'll be looking at in this section as well.

In the case of a cone the variable that sits by itself on one side of the equal sign will determine the axis that the cone opens up along. For instance, a cone that opens up along the *x*-axis will have the equation,

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x^2}{a^2}$$

For most of the following surfaces we will not give the other possible formulas. We will however acknowledge how each formula needs to be changed to get a change of orientation for the surface.

Cylinder
Here is the general equation of a cylinder.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is a cylinder whose cross section is an ellipse. If $a = b$ $a = b$ we have a cylinder whose cross section is a circle. We'll be dealing with those kinds of cylinders more than the general form so the equation of a cylinder with a circular cross section is,

$$x^2 + y^2 = r^2$$

Here is a sketch of typical cylinder with an ellipse cross section.

The cylinder will be centered on the axis corresponding to the variable that does not appear in the equation.

Be careful to not confuse this with a circle. In two dimensions it is a circle, but in three dimensions it is a cylinder.

Hyperboloid of One Sheet
Here is the equation of a hyperboloid of one sheet.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Here is a sketch of a typical hyperboloid of one sheet.

The variable with the negative in front of it will give the axis along which the graph is centered.

Hyperboloid of Two Sheets
Here is the equation of a hyperboloid of two sheets.

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Here is a sketch of a typical hyperboloid of two sheets.

The variable with the positive in front of it will give the axis along which the graph is centered.

Notice that the only difference between the hyperboloid of one sheet and the hyperboloid of two sheets is the signs in front of the variables. They are exactly the opposite signs.

Elliptic Paraboloid

Here is the equation of an elliptic paraboloid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

As with cylinders this has a cross section of an ellipse and if $a = b$ $a = b$ it will have a cross section of a circle. When we deal with these we'll generally be dealing with the kind that have a circle for a cross section.

In this case the variable that isn't squared determines the axis upon which the paraboloid opens up. Also, the sign of c will determine the direction that the paraboloid opens. If c is positive then it opens up and if c is negative then it opens down.

Hyperbolic Paraboloid

Here is the equation of a hyperbolic paraboloid.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$

These graphs are vaguely saddle shaped and as with the elliptic paraoloid the sign of c will determine the direction in which the surface "opens up". The graph above is shown for c positive.

With the both of the types of paraboloids discussed above the surface can be easily moved up or down by adding/subtracting a constant from the left side.